Calculus 140, section 4.8 Limits at Infinity

notes prepared by Tim Pilachowski

Recall from Lecture 2.2 – Definition of Limit: "Let f be a function defined at each point of some open interval containing a, except possibly a itself. Then a number L is the **limit of *f*(*x*) as *x* approaches *a*** (or is the **limit of *f* at *a***) if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$".

In section 2.3, we identified properties of limits that made the process much more expedient: Sum Rule, Constant Multiple Rule, Difference Rule, Product Rule, Quotient Rule and Substitution Rule.

In some of the examples done in chapter 2, we had $\lim\limits_{x \to a} f(x) = \infty$. Now we turn to consideration of $\lim\limits_{x \to \infty} f(x)$.

The text gives an ε definition of $\lim\limits_{x \to \infty} f(x) = L$ (Definition 4.15). Make sure that you're familiar with it.

Examples A: For the following functions, determine whether $\lim\limits_{x \to \infty} f(x) = L$ exists, and if so, the value of L.

a) $f(x) = 3$ b) $f(x) = \dfrac{3x^2 - 1}{2}$ c) $f(x) = \dfrac{3x^2 - 1}{2x - 1}$ d) $f(x) = \dfrac{3x^2 - 1}{2x^2 - 1}$ e) $f(x) = \dfrac{3x^2 - 1}{2x^3 - 1}$

answers: 3, ∞, ∞, $\frac{3}{2}$, 0

In the cases of Examples A-b and A-c, we obtained **infinite limits** (Definition 4.16).

In the cases of Examples A-d and A-e, we succeeded in identifying **horizontal asymptotes** (Definition 4.15) – $y = \frac{3}{2}$ for Example A-d and $y = 0$ for Example A-e.

Also note that the horizontal asymptotes of Examples A-d and A-e are in place both "forever left" and "forever right". That is, $\lim\limits_{x \to -\infty} \dfrac{3x^2 - 1}{2x^2 - 1} = \lim\limits_{x \to \infty} \dfrac{3x^2 - 1}{2x^2 - 1} = \dfrac{3}{2}$ and $\lim\limits_{x \to -\infty} \dfrac{3x^2 - 1}{2x^3 - 1} = \lim\limits_{x \to \infty} \dfrac{3x^2 - 1}{2x^3 - 1} = 0$.

Examples B: For the following functions, determine whether $\lim_{x\to\infty} f(x) = L$ exists, and if so, the value of L.

a) $f(x) = \ln(x-1)$ b) $f(x) = e^{x-1}$ c) $f(x) = e^{1-x}$ d) $f(x) = \dfrac{3}{2+e^{-x}}$ e) $f(x) = \cos(x-1)$

answers: ∞, ∞, 0, $\frac{3}{2}$, DNE

In the cases of Examples B-a and B-b, we obtained **infinite limits**.

In the cases of Examples B-c and B-d, we succeeded in identifying **horizontal asymptotes** –

$y = 0$ for Example B-c and $y = \frac{3}{2}$ for Example B-d.

Note that the horizontal asymptotes of Examples B-c and B-d are in place only "forever right". That is, while $\lim_{x\to\infty} e^{1-x} = 0$, $\lim_{x\to-\infty} e^{1-x} = \infty$, and while $\lim_{x\to\infty} \dfrac{3}{2+e^{-x}} = \dfrac{3}{2}$, $\lim_{x\to-\infty} \dfrac{3}{2+e^{-x}} = 0$.

A pair of basic but very important principles are illustrated in the Examples above: As a denominator → ∞, a fraction → 0, while any constants → themselves.

With respect to power functions, for any real number $n > 0$, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to\infty} \dfrac{1}{x^n} = 0$.

With respect to exponential functions, for any real number $k > 0$, $\lim_{x\to\infty} b^{kx} = \infty$ and $\lim_{x\to\infty} \dfrac{1}{b^{kx}} = 0$.

Finally, for a given constant a, $\lim_{x\to\infty} a = a$.

Examples C: For the following functions, determine the equations of any horizontal asymptotes.

a) $f(x)=\dfrac{\sin x}{x}$ b) $f(x)=\sqrt{x^2+x}-x$

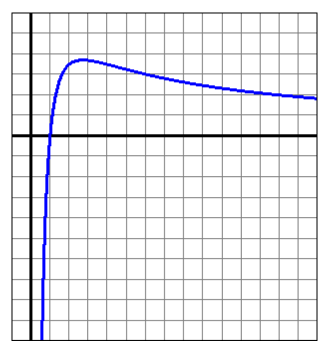
answers: $y = 0$, $y=\frac{1}{2}$

Examples D (4.5&7 Examples B and F revisited): Consider the functions

$f(x)=\dfrac{x^3}{e^x}$ and $g(x)=\dfrac{10\ln x}{x}$.

In both cases, finding the limit "forever left" yields the same result: $\lim\limits_{x\to-\infty}\dfrac{x^3}{e^x}=-\infty$ and $\lim\limits_{x\to-\infty}\dfrac{10\ln x}{x}=-\infty$.

However, (also in both cases), finding the limit "forever right" gives an indeterminate result: $\lim\limits_{x\to\infty}\dfrac{x^3}{e^x}=\dfrac{\infty}{\infty}$ and $\lim\limits_{x\to\infty}\dfrac{10\ln x}{x}=\dfrac{\infty}{\infty}$ that we cannot evaluate using the algebraic methods used in Examples A.



You won't have the calculus tools to evaluate this type of indeterminate limit until partway through Math 141.

We can approach these two limits in a different way to identify the horizontal asymptote as being $y = 0$ in both cases.

Once we get to the right of the absolute maximum, we know two things.

1) Both $f(x)$ and $g(x)$ are decreasing functions.

2) Both $f(x)$ and $g(x)$ are "positive numerator over positive denominator", so their y-coordinates are positive.

We conclude that, "forever to the right", both graphs approach $y = 0$ without ever touching it.

For both functions, $y = 0$ is a horizontal asymptote in place on the right.